New coherent states and inhomogeneous differential realization of Lie superalgebra B(0,1)

1993 J. Phys. A: Math. Gen. 26 913

(http://iopscience.iop.org/0305-4470/26/4/018)

J. Phys. A: Math. Gen. 26 (1993) 913-919. Printed in the UK

New coherent states and inhomogeneous differential realization of Lie superalgebra $B(0, 1)$

Le-Man Kuang and Gao-Jian Zeng

CCAST (World Laboratory), PO Box 8730, Beijing, People's Republic of China, and mailing address, Department of Physics, Hunan Normal University, Hunan 410006, People's Republic of China

Received 15 April 1992, in final form 14 September 1992

Abstract. A new kind of coherent states for the Lie superalgebra $B(0, 1)$ is presented. An inhomogeneous differential realization (THDR) of the Lie superalgebra is obtained. This IHDR may be useful for the study of the quasi-exactly solvable problems of quantum mechanics.

1. Introduction

It is well known that coherent states [1] of Lie (super)algebras have played an important role in the study of quantum mechanics systems. The coherent states of quantum optics [2], which provide a natural link between classical and quantum mechanics and are related to the path integral formalism, were extended to arbitrary Lie groups by Perelomov and Gilmore [3, 4]. Generalized coherent states for a Lie group G are defined by acting with some irrep of G on a fixed vector carrying a one-dimensional irrep of a subgroup H. Later on, they are further extended to deal with finite-dimensional vector representations of H [5-7]. Recently, generalized coherent states for Lie superalgebras have begun to be investigated by many authors [8-10].

Recently discovered quasi-exactly solvable problems (QESP) [11-14] of quantum mechanics become more and more important because they have been generalized, in parallel, to study the conformal field theory [15]. The QESP have been proved to be related to the inhomogeneous differential realizations (IHDR) of Lie (super)algebras [12-14, 16]. Turbiner gave a complete classification of the one-dimensional QESP by making use of the IHDR of the SU(2) algebra, and pointed out that one can study the multi-dimensional QESP [12]. Shifman and Turbiner presented the general procedure to construct the multi-dimensional QESP by making use of the IHDR of the Lie superalgebras [13, 14]. Hence it is very important to study the IHDR of Lie superalgebras.

In this paper, by analysing the properties of the finite-dimensional irrep of the Lie superalgebra $B(0, 1)$ we present a new kind of coherent states for the $B(0, 1)$ superalgebra. We also study the IHDR of the superalgebra in the new coherent state space.

2. *New coherent states for the $B(0, 1)$ superalgebra*

The Lie superalgebra $B(0, 1)$ [17] consists of three even generators $L_{\pm}$, L_3 and two odd generators $V_{\pm}$ which satisfy the following commutation and anticommutation

0305-4470/93/040913+07$7.50

relations

$$[L_3, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = -L_3$$
$$[L_3, V_\pm] = \pm V_\pm \qquad [L_+, V_+] = 0 \qquad [L_\pm, V_\mp] = \pm V_+/\sqrt{2}. \tag{1}$$
$$\{V_\pm, V_\pm\} = -L_\pm/\sqrt{2} \qquad \{V_+, V_-\} = -L_3$$

The basis vectors of a finite-dimensional irrep of the superalgebra may be labelled $|J, M, \alpha\rangle$, where $\alpha = 0$ and 1, $J = 0, \frac{1}{2}, 1, \ldots,$ and

$$M = \begin{cases} -J, -J+1, \ldots, J, & \text{for } \alpha = 0 \\ -J, -J+1, \ldots, J-1, & \text{for } \alpha = 1. \end{cases} \tag{2}$$

With a convenient normalization of the basis vectors, the way in which the generators act upon the basis vectors is indicated by the relations:

$$\begin{aligned} L_3|J, M, \alpha\rangle &= (M + \tfrac{1}{2}\alpha)|J, M, \alpha\rangle \\ L_+|J, M, \alpha\rangle &= -[(J+M+1)(J-M-\alpha)]^{1/2}/\sqrt{2}|J, M+1, \alpha\rangle \\ L_-|J, M, \alpha\rangle &= [(J+M)(J-M+1-\alpha)]^{1/2}/\sqrt{2}|J, M-1, \alpha\rangle \\ V_+|J, M\,\alpha\rangle &= -(1-\alpha)(J-M-\alpha)^{1/2}|J, M, \alpha+1\rangle - \alpha(J+M+1)^{1/2}|J, M, \alpha-1\rangle \\ V_-|J, M, \alpha\rangle &= (1-\alpha)(J+M)^{1/2}|J, M-1, \alpha+1\rangle - \alpha(J-M-\alpha+1)^{1/2}|J, M, \alpha-1\rangle. \end{aligned} \tag{3}$$

This irrep is $4J+1$ dimensional and may be split as the two subspaces $|J, M, 0\rangle$ and $|J, M, 1\rangle$ corresponding to $\alpha = 0$ and 1, respectively. The completeness condition of the basis vectors of the irrep may be expressed as

$$\sum_{n=0}^{2J} |J, -J+n, 0\rangle\langle J, -J+n, 0| + \sum_{n=0}^{2J-1} |J, -J+n, 1\rangle\langle J, -J+n, 1| = I \tag{4}$$

where I is the identity operator.

From (3) one may obtain two useful formulas for computation

$$\begin{aligned} L_+^n|J, -J, 0\rangle &= \left(-\frac{1}{\sqrt{2}}\right)^n \binom{2J}{n}^{1/2} n!|J, -J+n, 0\rangle \\ L_+^n|J, -J, 1\rangle &= \left(-\frac{1}{\sqrt{2}}\right)^n \binom{2J-1}{n}^{1/2} n!|J, -J+n, 1\rangle. \end{aligned} \tag{5}$$

We now introduce two coherent states $|z\rangle_1$ and $|z\rangle_2$ by applying the exponential operator $\exp(zL_+)$ on the lowest-weight states $|J, -J, 0\rangle$ and $|J, -J, 1\rangle$ of the two subspaces of a $B(0, 1)$ irrep

$$|z\rangle_1 = N_1(z)\exp(zL_+)|J, -J, 0\rangle \tag{6a}$$

$$|z\rangle_2 = N_2(z)\exp(zL_+)|J, -J, 1\rangle \tag{6b}$$

where $N_1(z)$ and $N_2(z)$ are two normalization constants to be determined.

Making use of equation (5), one can rewrite (6) as follows:

$$|z\rangle_1 = N_1(z) \sum_{n=0}^{2J} \binom{2J}{n}^{1/2} \left(-\frac{z}{\sqrt{2}}\right)^n |J, -J+n, 0\rangle \tag{7a}$$

$$|z\rangle_2 = N_2(z) \sum_{n=0}^{2J-1} \binom{2J-1}{n}^{1/2} \left(-\frac{z}{\sqrt{2}}\right)^n |J, -J+n, 1\rangle. \tag{7b}$$

The analysis below shows that the new coherent states of the Lie superalgebra $B(0,1)$ consist of the two subsystems $|z\rangle_2$ and $|z\rangle_2$, which are related to the two subspaces $\{|J,M,0\rangle\}$ and $\{|J,M,1\rangle\}$ of the $B(0,1)$ irrep. We use the symbol $\{|z\rangle_1, |z\rangle_2\}$ to denote the new coherent states in which $\{|z\rangle_1\}$ and $\{|z\rangle_2\}$ can be regarded as two subspaces.

We require that the new coherent states are normalized in the form

$$ {}_i\langle z|z\rangle_i = 1 \qquad (i=1,2). \tag{8} $$

It follows from (7) and (8) that

$$ N_1(z) = (1+\tfrac{1}{2}z\bar{z})^{-J} \qquad N_2(z) = (1+\tfrac{1}{2}z\bar{z})^{-J+1/2} \tag{9} $$

where $\bar{z}$ is the complex conjugation of z.

The scalar product of two new coherent states is of the form

$$ {}_1\langle z'|z\rangle_1 = N_1(z')N_1(z)(1+\tfrac{1}{2}z\bar{z}')^{2J} \tag{10a} $$

$$ {}_2\langle z'|z\rangle_2 = N_2(z')N_2(z)(1+\tfrac{1}{2}z\bar{z}')^{2J-1} \tag{10b} $$

$$ {}_1\langle z'|z\rangle_2 = 0. \tag{10c} $$

which indicate that two new coherent states in the same subspace are non-orthogonal. Nevertheless, two coherent states in different subspaces are orthogonal to each other.

Making use of the orthogonality of the basis vectors of the $B(0,1)$ irrep, from (6) we find the expansion coefficients

$$ \begin{aligned} \langle J,M,0|z\rangle_1 &= N_1(z)\binom{2J}{J+M}^{1/2}\left(-\frac{z}{\sqrt{2}}\right)^{J+M} \qquad \langle J,M,0|z\rangle_2 = 0 \\ \langle J,M,1|z\rangle_2 &= N_2(z)\binom{2J-1}{J+M}^{1/2}\left(-\frac{z}{\sqrt{2}}\right)^{J+M} \qquad \langle J,M,1|z\rangle_1 = 0. \end{aligned} \tag{11} $$

We now study the completeness condition of the new coherent states. Since the $4J+1$ state vectors $\{|J,M,\alpha\rangle\}$ of an irrep of the $B(0,1)$ superalgebra are known to form a completeness orthogonal set, the problem here may be changed to find the following two weight functions $\sigma_1(z)$ and $\sigma_2(z)$ such that

$$ \begin{aligned} &\int \mathrm{d}\sigma_1(z)|z\rangle_{11}\langle z| + \int \mathrm{d}\sigma_2(z)|z\rangle_{22}\langle z| \\ &\quad = \sum_{n=0}^{2J} |J,-J+n,0\rangle\langle J,-J+n,0| + \sum_{n=0}^{2J-1} |J,-J+n,1\rangle\langle J,-J+n,1| = I. \end{aligned} \tag{12} $$

Let $|f\rangle$ and $|g\rangle$ be two arbitrary vectors, then equation (12) means that

$$ \langle f|g\rangle = \int \mathrm{d}\sigma_1(z)\langle f|z\rangle_{11}\langle z|g\rangle + \int \mathrm{d}\sigma_2(z)\langle f|z\rangle_{22}\langle z|g\rangle. \tag{13} $$

We now determine the two weight functions. Let

$$ \mathrm{d}\sigma_1(z) = \sigma_1(r)r\,\mathrm{d}r\,\mathrm{d}\theta \tag{14a} $$

$$ \mathrm{d}\sigma_2(z) = \sigma_2(r)r\,\mathrm{d}r\,\mathrm{d}\theta \tag{14b} $$

where we have set $z = r\,\mathrm{e}^{\mathrm{i}\theta}$.

Substituting the definition (7) into (13) and integrating over the variable θ, we have

$$\langle f|g\rangle=\int_0^\infty r\sigma_1(r)\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\sum_{n,m=0}^{2J}\binom{2J}{n}^{1/2}\binom{2J}{m}^{1/2}z^m\bar{z}^n$$
$$\times(1+\tfrac{1}{2}z\bar{z})^{-2J}\langle f|J,-J+m,0\rangle\langle J,-J+n,0|g\rangle$$
$$+\int_0^\infty r\,\mathrm{d}\sigma_2(r)\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\sum_{n,m=0}^{2J-1}\binom{2J-1}{n}^{1/2}\binom{2J-1}{m}^{1/2}z^m\bar{z}^n(1+\tfrac{1}{2}z\bar{z})^{-2J+1}$$
$$\times\langle f|J,-J+m,1\rangle\langle J,-J+n,1|g\rangle \tag{15}$$
$$=2\pi\sum_{n=0}^{2J}\binom{2J}{n}\int_0^\infty r^{2n+1}\sigma_1(r)(1+\tfrac{1}{2}r^2)^{-2J}\,\mathrm{d}r$$
$$\times\langle f|J,-J+n,0\rangle\langle J,-J+n,0|g\rangle$$
$$+2\pi\sum_{n=0}^{2J-1}\binom{2J-1}{n}r^{2n+1}\sigma_2(r)(1+\tfrac{1}{2}r^2)^{-2J+1}\,\mathrm{d}r$$
$$\times\langle f|J,-J+n,1\rangle\langle J,-J+n,1|g\rangle. \tag{16}$$

Hence we must have

$$2\pi\binom{2J}{n}\int_0^\infty r^{2n+1}(1+\tfrac{1}{2}r^2)^{-2J}\sigma_1(r)=1 \tag{17}$$

$$2\pi\binom{2J-1}{n}\int_0^\infty r^{2n+1}(1+\tfrac{1}{2}r^2)^{-2J+1}\sigma_2(r)=1. \tag{18}$$

With the help of the following integral formula

$$\int_0^\infty\frac{x^{2n+1}}{(1+x^2)^m}\,\mathrm{d}x=\frac{n!(m-n-2)!}{2(m-1)!} \tag{19}$$

comparing equations (17) and (18) with (19) we finally obtain the desired weight functions

$$\sigma_1(r)=\frac{2J+1}{\sqrt{2}\,\pi(1+\tfrac{1}{2}r^2)^2} \tag{20}$$

$$\sigma_2(r)=\frac{2J}{\sqrt{2}\,\pi(1+\tfrac{1}{2}r^2)^2}. \tag{21}$$

Thus, the completeness relation of the new coherent states can be written as

$$\frac{1}{\sqrt{2}\,\pi}\int\frac{\mathrm{d}^2z}{(1+\tfrac{1}{2}z\bar{z})^2}[(2J+1)|z\rangle_{11}\langle z|+2J|z\rangle_{22}\langle z|]=1. \tag{22}$$

As a result of the above completeness condition, an arbitrary vector $|\psi\rangle$ can be expanded in terms of the new coherent states for the $B(0,1)$ superalgebra as follows:

$$|\psi\rangle=\frac{1}{\sqrt{2}\,\pi}\int\frac{\mathrm{d}^2z}{(1+\tfrac{1}{2}z\bar{z})^2}[(2J+1)|z\rangle_{11}\langle z|\psi\rangle+2J|z\rangle_{22}\langle z|\psi\rangle]. \tag{23}$$

3. An inhomogeneous differential realization for the $B(0,1)$ superalgebra

In this section, we study the IHDR of the $B(0,1)$ superalgebra in the new coherent state space. For simplicity, we consider the IHDR of the superalgebra in the unnormalized new coherent state space $\{\|z\rangle_1, \|z\rangle_2\}$ defined by

$$\|z\rangle_1=\sum_{n=0}^{2J}\binom{2J}{n}^{1/2}\left(-\frac{z}{\sqrt{2}}\right)^n|J,-J+n,0\rangle \tag{24a}$$

$$\|z\rangle_2=\sum_{n=0}^{2J-1}\binom{2J-1}{n}^{1/2}\left(-\frac{z}{\sqrt{2}}\right)^n|J,-J+n,1\rangle. \tag{24b}$$

We now consider the actions of the $B(0,1)$ generators on $\{\|z\rangle_1, \|z\rangle_2\}$. First we calculate the action of L_- on the subspace $\{\|z\rangle_1\}$. Making use of the completeness relation (4) and the expansion coefficients (11) and (3), we have

$$\begin{aligned}L_-\|z\rangle_1&=\sum_{n=0}^{2J}L_-|J,-J+n,0\rangle\langle J,-J+n,0\|z\rangle_1+\sum_{n=0}^{2J}L_-|J,-J+n,1\rangle\langle J,-J+n,1\|z\rangle_1\\&=-\tfrac{1}{2}z\sum_{n=0}^{2J}\binom{2J}{n}^{1/2}(2J-n)\left(-\frac{z}{\sqrt{2}}\right)^n|J,-J+n,0\rangle\\&=-\tfrac{1}{2}z(2J-z\,\mathrm{d}/\mathrm{d}z)\|z\rangle_1\end{aligned} \tag{25}$$

which indicates that the generator L_- acts like an inhomogeneous differential operator l_-^{11} on the subspace $\{\|z\rangle_1\}$,

$$l_-^{11}=-\tfrac{1}{2}z(2J-z\,\mathrm{d}/\mathrm{d}z). \tag{26}$$

In the same way, one may obtain the action of L_- on the second subspace $\{\|z\rangle_2\}$:

$$L_-\|z\rangle_2=l_-^{22}\|z\rangle_2 \tag{27}$$

with

$$l_-^{22}=-\tfrac{1}{2}z(2J-1-z\,\mathrm{d}/\mathrm{d}z). \tag{28}$$

Then the action of the generator L_- on the new coherent states $\{\|z\rangle_1, \|z\rangle_2\}$ can be expressed as

$$L_-\begin{pmatrix}\|z\rangle_1\\ \|z\rangle_2\end{pmatrix}=\rho(L_-)\begin{pmatrix}\|z\rangle_1\\ \|z\rangle_2\end{pmatrix} \tag{29}$$

where the matrix operator $\rho(L_-)$ is given by

$$\rho(L_-)=\begin{pmatrix}l_-^{11} & 0\\ 0 & l_-^{22}\end{pmatrix}. \tag{30}$$

Similarly, one can get the actions of the other generators on the new cohrent states $\{\|z\rangle_1, \|z\rangle_2\}$:

$$\rho(L_+)=\begin{pmatrix}l_+^{11} & 0\\ 0 & l_+^{22}\end{pmatrix}\qquad \rho(L_3)=\begin{pmatrix}l_3^{11} & 0\\ 0 & l_3^{22}\end{pmatrix} \tag{31}$$

$$\rho(V_+)=\begin{pmatrix}0 & v_+^{12}\\ v_+^{21} & 0\end{pmatrix}\qquad \rho(V_-)=\begin{pmatrix}0 & v_-^{12}\\ v_-^{21} & 0\end{pmatrix} \tag{32}$$

where these matrix elements in equations (31) and (32) are given by

$$l_+^{11} = -\frac{z}{\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}z} \qquad l_+^{22} = -\frac{z}{\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}z} \tag{33}$$

$$l_3^{11} = -J + z\frac{\mathrm{d}}{\mathrm{d}z} \qquad l_3^{22} = -J + \tfrac{1}{2} + z\frac{\mathrm{d}}{\mathrm{d}z} \tag{34}$$

$$v_+^{12} = -\tfrac{1}{2}(2J)^{1/2} \qquad v_+^{21} = \frac{\sqrt{2}}{2}(2J)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}z} \tag{35}$$

$$v_-^{12} = -\tfrac{1}{4}\sqrt{2}\, z(2J)^{1/2} \qquad v_-^{21} = -\tfrac{1}{2}(2J)^{-1/2}\left(2J - z\frac{\mathrm{d}}{\mathrm{d}z}\right). \tag{36}$$

It can be shown that these matrix operators from (30) to (32) satisfy the commutation and anticommutation relations of the Lie superalgebra $B(0,1)$ when they act upon the new coherent state space $\{\|z\rangle_1, \|z\rangle_2\}$. Therefore, they give rise to an IHDR of the $B(0,1)$ superalgebra.

4. Concluding remarks

We have constructed a new kind of coherent states for the $B(0,1)$ superalgebra, and obtained the IHDR of the superalgebra. The new coherent states have some advantages such as they contain only one usual complex variable without any Grassmann variables, so that one need not make calculations on Grassmann variables when the new coherent states are applied to concrete physical problems. In the new coherent states space the generators of the Lie superalgebra can be realized in the matrix form, and this realization is an IHDR which is just needed in the QESP.

It is possible to generalize this kind of new coherent states to other Lie superalgebras. A straightforward example is the SPL(2, 1) superalgebra [18]. A finite-dimensional irrep of the SPL(2, 1) superalgebra may be split as the four subspaces, so one can define four coherent states which together form the new coherent states of the SPL(2, 1) superalgebra.

It is also interesting to extend this formalism to quantum superalgebras.

References

[1] Klauder J R and Skagerstam B S (ed) 1985 *Coherent States* (Singapore: World Scientific)
[2] Glauber R J 1963 *Phys. Rev.* **130** 2529; *Phys. Rev.* **131** 2766
[3] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222; 1975 *Commun. Math. Phys.* **44** 197; 1977 *Sov. Phys. Usp.* **20** 703
[4] Gilmore R 1972 *Ann. Phys.* **74** 391
Gilmore, R, Bowden C M and Narducci L M 1975 *Phys. Rev.* A **12** 1019
[5] Deenen J and Quesne C 1984 *J. Math. Phys.* **25** 2354; 1985 *J. Math. Phys.* **26** 2705
[6] Rowe D J, Rosensteel G and Gilmore R 1985 *J. Math. Phys.* **26** 2787
[7] Quesne C 1986 *J. Math. Phys.* **27** 428, 869
[8] Quesne C 1990 *J. Phys. A: Math. Gen.* **23** L43; 1990 *J. Phys. A: Math. Gen.* **23** 5383, 5411
[9] Balantekin A B, Schmitt H A and Barrett B R 1988 *J. Math. Phys.* **29** 1634
Schmitt H A and Mufti 1990 *J. Phys. A: Math. Gen.* **23** L861; 1991 *J. Phys. A: Math. Gen.* **24** L815
[10] Blanc L and Rowe D J 1989 *J. Math. Phys.* **30** 1415; 1990 *J. Math. Phys.* **31** 14
[11] Turbiner A V and Ushiveridze A G 1987 *Phys. Lett.* **126A** 181

[12] Turbiner A V 1988 *Commun. Math. Phys.* **118** 467
[13] Shifman M A and Turbiner A V 1989 *Commun. Math. Phys.* **126** 347
[14] Shifman M A 1988 New findings in quantum mechanics *Preprint* CERN-TH 5265/88
[15] Morozov A Y, Perelomov A M and Rosly A A 1990 *Int. J. Mod. Phys.* **5A** 803
[16] Fu H C and Sun C P 1990 *J. Math. Phys.* **31** 287; 1991 *J. Math. Phys.* **32** 767
[17] Fu H C 1990 *High Energy Phys. Nucl. Phys.* **14** 126 (in Chinese) (to appear in American edition)
[18] Scheunert M, Nahm W and Rittenberg V 1977 *J. Math. Phys.* **18** 155